# A Fuzzy Description Logic with Hedges as Concept Modifiers

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**Abstract:** In this paper we present a fuzzy description logic $\mathcal{ALC}_{FH}$, where primitive concepts are modified by means of hedges. $\mathcal{ALC}_{FH}$ is strictly more expressive than Fuzzy-$\mathcal{ALC}$ defined in [8]. We show that given a linearly ordered set of hedges primitive concepts can be modified to any desired degree by prefixing them with appropriate chains of hedges. Furthermore, we define a decision procedure for the unsatisfiability problem in $\mathcal{ALC}_{FH}$, and discuss truth bounds, expressivity as well as complexity issues.

**Key words:** fuzzy logic, hedge algebras, description logic

## 1. Introduction

In many areas of computer science and artificial intelligence logic has been accepted as the mathematical foundation for knowledge representation and reasoning. Beyond that it has been recognized that the logic itself can be used as computation unifying in an almost ideal way declarative and operational semantics. Among the many applications description logics have been particularly successful (see e.g. $[1, 2]$). The main object of description logic is to fix a terminology by describing concepts and their relations, and to provide (decidable) key services for reasoning wrt this terminology like, for example, subsumption and unsatisfiability testing.

In all applications of description logis that we are aware of except [8] and [9] concepts are crisp unary relations, i.e., an object may or may not be an element of a particular concept. On the other hand, in many real-world applications like, for example, intelligent e-commerce information is often vague and imprecise. We may observe that a customer is interested in technical aspects, whereas he or she is not interested in design issues. Fuzzy set theory introduced by Lotfi A. Zadeh (see e.g. [10]) provides an ability to denote non-crisp concepts, i.e., an object may belong to a certain degree (typically a real number from the interval $[0, 1]$) to a particular relation. For instance, the semantic content of custumer Robert being interested in technical aspects may be described by means of a statement like “Robert is interested in technical aspects” and establishing that this sentence has a degree, or truth–value, of $0.8$.

Humans typically use linguistic adverbs like “very”, “more or less” etc. to distiguish, for example, between a custumer who is interested in technical details and one who is very interested in these details. In [6] Zadeh introduces so-called linguistic hedges modifying the shape of a fuzzy set by transforming it into another. For instance, he introduced the operator *CON* (for “contraction”) that maps the membership function $\mu_A$ of a fuzzy set $A$ to a membership function $\mu_{very\,A}$ that has high degrees of membership only for those elements of the domain, that belong “very much” to $A$. In other words, the degree of membership for elements that already had a low degree of membership in $A$ is decreased, whereas the degree of membership for elements with a high degree of membership is left high. Technically, Zadeh achieves this by simply raising the degree of membership to the $\beta$-th power, where $\beta > 1$ is a constant which can be fixed by the application. For example, $\mu_{very\,A}(u) = \mu_A(u)^2$. This technical solution calls for the introduction of an opposite operator *DIL* (for dilation) that maps the degree of membership to the $\beta'$-th power, where $0 < \beta' < 1$, thereby strengthening the degree of membership of elements with small degree. One should note that these hedges can be concatenated, for example, if we want to consider elements that belong to $A$ “very very much”, then $\mu_A$ can be transformed into its $\beta^2$-th power by applying the operator *CON* twice: *CON CON* $A$.

Because the gap between $A$ and *CON* $A$ in this setting is quite large, it seems a good idea to refine this idea. In many human languages there is almost a continuum of phrases like “more or less”, “much less”, “possibly rather” and so forth expressing different levels of emphasis. The so called *hedge algebras* introduced in [4] give an algebraic structure that formally defines such hedges and structures their relationships. They have been applied to fuzzy logic in various ways (see e.g. [3]), but to our knowledge, hedge algebras have not been applied as concept modifiers in description logic.

In this paper we apply linear ordered hedge algebras as concept modifiers in the framework of a fuzzy description logic. The paper is organized as follows: In the following Section 2. we recall basic notions from fuzzy description logics and the structure of hedges. In particular, we define an extended fuzzy description logic with hedges $\mathcal{ALC}_{FH}$. Con-

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cept modifiers are formally introduced in Section 3.. A decision procedure for the unsatisfiability problem of $\mathcal{A}LC_{FH}$ is presented in Section 4.. Expessivity and complexity issues are discussed in Sections 5. and 6. Finally, we discuss our findings in Section 7.

# 2. Logical Basics

## 2.1. Fuzzy Description Logic

Concepts are expressions that collect the properties, described–among others–by means of roles of a set of individuals. From a first order logical viewpoint, concepts can be seen as unary predicates, whereas roles are interpreted as binary predicates. A concept can be build out of primitive concepts, roles, modifiers and combinations of other concepts (see Table 1). For the fuzzy extension, a concept $C$, rather than being interpreted as a classical set, will be represented as a fuzzy set and, thus, concepts become imprecise. Consequently, a statement like "$a$ is $C$", where $a$ is an individual, will have a truth-value in $[0, 1]$ denoting the degree of the membership of $a$ in the fuzzy set $C$.

| | | |
|---|---|---|
| $C, D \rightarrow$ | $A$ | (primitive concept) |
| | $R$ | (primitive role) |
| | $\top$ | (top concept) |
| | $\bot$ | (bottom concept) |
| | $\neg C$ | (negation) |
| | $MA$ | (concept modification) |
| | $C \sqcap D$ | (concept conjunction) |
| | $C \sqcup D$ | (concept conjunction) |
| | $\forall R.C$ | (universal quantification) |
| | $\exists R.C$ | (existential quantification) |

Table 1: The language $\mathcal{A}LC_{FH}$ of the description logic $\mathcal{A}LC$ extended by fuzzy hedges. As a notational convention, $C$ and $D$ denote concepts, $A$ primitive concepts, $R$ roles and $M$ concept modifiers, all of them possibly with subscripts.

The semantics is based on the notion of an interpretation. An interpretation is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of an non-empty set $\Delta^{\mathcal{I}}$ (called the domain) and an interpretation function $\cdot^{\mathcal{I}}$ mapping individuals to elements of $\Delta^{\mathcal{I}}$, concepts $C$ to a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$, and roles $R$ to a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$. Therefore, if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$, then $C^{\mathcal{I}}(d)$ is the degree of $d$ being an element of the fuzzy concept $C$ under $\mathcal{I}$. Roles are interpreted in the same way. The interpretation for complex concepts is defined in Table 2.[3]

Two concepts $C$ and $D$ are said to be *equivalent* (denoted by $C \cong D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretations $\mathcal{I}$. For instance, $\top \cong \neg\bot$, $(C \sqcap D) \cong \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong \neg(\exists R.\neg C)$.

[3] The similarity of the definitions of many of the operators defined in Table 2 with fuzzy set operations [5] is intended.

$$
\begin{aligned}
A^{\mathcal{I}} &: \Delta^{\mathcal{I}} \rightarrow [0,1] \\
R^{\mathcal{I}} &: \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1] \\
\top^{\mathcal{I}}(d) &= 1 \text{ for all } d \in \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}}(d) &= 0 \text{ for all } d \in \Delta^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}}(d) &= \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) &= \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(\neg C)^{\mathcal{I}}(d) &= 1 - C^{\mathcal{I}}(d) \\
(MA)^{\mathcal{I}}(d) &= \eta_M(A^{\mathcal{I}}(d)) \\
(\forall R.C)^{\mathcal{I}}(d) &= \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d')\}\} \\
(\exists R.C)^{\mathcal{I}}(d) &= \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d')\}\}
\end{aligned}
$$

Table 2: Interpretations of $\mathcal{A}LC_{FH}$. $\eta_M$ is a membership modifier discussed later.

A *fuzzy assertion* (denoted by $\psi$) is one of $\langle \alpha > n \rangle$, $\langle \alpha \geq n \rangle$, $\langle \alpha \leq n \rangle$, $\langle \alpha < n \rangle$ and $\langle \alpha = n \rangle$ where $\alpha$ is an expression of type $a : C$ ("$a$ is in $C$"), $(a, b) : R$ ("$(a, b)$ is in $R$") and $n \in [0, 1)$. From a semantical viewpoint, a fuzzy assertion $\langle \alpha \geq n \rangle$ constrains the truth-value of $\alpha$ to be greather or equal to $n$. Formally, an interpretation $\mathcal{I}$ satisfies a fuzzy assertion $\langle a : C \geq n \rangle$ (resp. $\langle (a, b) : R \geq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$). The operators $>, =, <$ and $\leq$ are interpreted in the same way. Two fuzzy assertion $\psi_1$ and $\psi_2$ are said to be *equivalent* (denoted by $\psi_1 \cong \psi_2$) iff they are satisfied by the same set of interpretations.

A *fuzzy terminological axiom* is either a *fuzzy concept specialization* of the form $A :\leq C$ or a *fuzzy concept definition* of the form $A := C$. A specialisation denotes the "more specific than"-relation between concepts, and a definition denotes the equivalence of concepts. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy concept specialisation $A :\leq C$ iff for all $d \in \Delta^{\mathcal{I}}$ we find $A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d)$, whereas $\mathcal{I}$ satisfies a fuzzy concept definition $A := C$ iff for all $d \in \Delta^{\mathcal{I}}$ we find $A^{\mathcal{I}}(d) = C^{\mathcal{I}}(d)$.

A *fuzzy knowledge base* $\Sigma$ is the union of a finite set $\Sigma_A$ of fuzzy assertions and a finite set $\Sigma_T$ of fuzzy terminological axioms. An interpretation $\mathcal{I}$ *satisfies* (or is a *model* of) a fuzzy knowledge base $\Sigma$ iff $\mathcal{I}$ satisfies each element of $\Sigma$. A fuzzy knowledge base *entails* a fuzzy assertion $\psi$ (denoted by $\Sigma \models \psi$) iff every model of $\Sigma$ also satisfies $\psi$. Furthermore, let $C$ and $D$ be two concepts, $D$ *fuzzy subsumes* $C$ with respect to $\Sigma_T$ (denoted by $C \preceq_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of $\Sigma$ and for all $d \in \Delta^{\mathcal{I}}$ we find $A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d)$.

The problem of determining whether $\Sigma \models \psi$ is called *entailment problem*, whether $C \preceq_{\Sigma_T} D$ is called *fuzzy subsumption problem*, and whether $\Sigma$ is satisfiable is called *satisfiability problem*.

### 2.2. The Structure of Hedges

In fuzzy set theory, hedges ware defined as unary operators which act on fuzzy sets, and form new fuzzy sets [6], for instance, $\mu_{very A}(u) = \mu_A(u)^2$, where $u$ an element of the universe $U$ and $\mu_A$ is the membership function of the fuzzy set $A$. Starting from a set of hedges $H$, for example $\{$*very* , *mol* , *rather* , *possibly* $\}$,[4] a hedge algebra defines an ordering relation $\leq$ on linguistic terms. In the work of Nguyen Cat Ho *etal*. [4, 3] linguistic terms are defined as chains of hedges applied to so-called generators, whose truth value is modified by the hedges. In the context of description logics the generators are the primitive concepts. In this paper, we assume that $H$ is linearly ordered such that each two hedges are comparable. In the literature hedge algebras with partially ordered sets of hedges are also studied by extending refined hedge algebras with distributive properties. In principle, the method for mapping linguistic terms to fuzzy sets seems to be applicable in case of partially ordered sets of hedges as well if a linearization of the order is considered.

In contrast to the hedges introduced by Zadeh, a chain of hedges has a meaning only taken as a whole: each new hedge put in front refines the meaning of the others. Each hedge is either positive or negative wrt the others, i.e., it increases or decreases the meaning, respectively. ”Very”, for example, can be positive to ”very” when it strenghtens the effect: $x \leq$ *very* $x \leq$ *very very* $x$. On the other hand, it can be at the same time negative to other hedges like ”possibly”, when it reduces its effect: *possibly* $x \leq$ *very possibly* $x \leq x$, and, thus, *very possibly* $x$ is less different from the original meaning $x$ than *possibly* $x$.

To discuss hedge algebras extensively is out of the scope of the paper. Thus, we will only introduce those notions and properties that will be used in the remainder of the paper. Two lingistic terms $u$ and $v$ are *independent* if neither of them can be formed by putting some more hedges in front of the other. To capture this property formally, we introduce $H(x)$ as the set of linguistic terms that can be formed by prefixing $x$ with chains of hedges: $H(x) = \{\delta x \mid \delta \text{ is a chain of hedges}\}$. Then, $u$ and $v$ are independent if $u \notin H(v)$ and $v \notin H(u)$. Consequently we have that $x \notin H(v)$ for all $x \in H(u)$. For two different hedges $h$ and $k$, prefixing other hedges $h'$ and $k'$ to $h$ and $k$ resp. can not change the semantical ordering relationship between $hx$ and $kx$, i.e. $h'hx < k'kx$ iff $hx < kx$.

Representating a concept as a fuzzy set we can use hedge properties for concept modifiers. Then, from a fuzzy assertion $\psi$ like $\langle a : A = n\rangle$ we may infer that $\langle a : \textit{very}\, A = n^{\beta}\rangle$, with $\beta > 1$, and $\langle a : \textit{mol}\, A = n^{\beta'}\rangle$, with $0 < \beta' < 1$.

In the next section we will define concept manipulators using hedge algebras

[4] *mol* is an abbreviation for *more or less* .

## 3. Concept Modifiers

Let us consider the set $H = \{h_1, h_2, ..., h_p\}$ of hedges, among which each element of $H$ is either positive or negative wrt all primitive concepts and all other hedges including itself. Let $\Omega$ be the set of all primitive concepts, we define *sign* $: H \times (H \cup \Omega) \rightarrow \{-1, 1\}$ with

$$\textit{sign}\,(h_i, h^*) = \begin{cases} -1 & \text{if } h_i \text{ is negative wrt } h^*, \\ 1 & \text{if } h_i \text{ is positive wrt } h^*. \end{cases}$$

where $h_i \in H$, $h^* \in H \cup \Omega$. The mapping *sign* tells us whether $h_i$ increases or decreases the meaning of $h^*$, if $h^* \in \Omega$, or $h^*\delta A$, where $h^*$ is a hedge, $A$ is a primitive concept and $\delta$ is any chain of hedges.

In our extension of Fuzzy-$\mathcal{ALC}$ we introduce concept modifiers $M$ in form of a chain of hedges $M = k_q k_{q-1} \ldots k_1$ with $k_i \in H$, for all $i = \overline{1, q}$. In this section we will define a mapping $\eta_M : [0, 1] \rightarrow [0, 1]$ that gives each hedge chain a meaning within the interpretation defined in Table 2. We will follow the idea of Zadeh to use power functions for this purpose.

**Definition 1.** A membership modifier is an exponential function $\eta : [0, 1] \rightarrow [0, 1]$ with $\eta(x) = x^{\beta}$, where $\beta > 0$.

So we have to calculate an exponent $\beta$ for each sequence of hedges. In the following, we will present an algorithm *exponent* which calculates $\beta$ for each hedge chain from a giving hedge set $H$ representing a membership modifier based on following ideas:

- We extend the idea behind Zadeh's operators *CON* and *DIL* discussed in the introduction in that we keep the idea of taking a power function with exponents $\beta > 1$ for positive and $0 < \beta < 1$ for negative hedges, but vary the exponents depending on the hedge. For instance, if the degree of being 25 years “young” is $0.8$, then the degree of being 25 years “very young” might be $0.8^2 = 0.64$, whereas 25 years “more or less young” may have the degree of $0.8^{1/2} = 0.894$ .
- Membership modifiers should preserve the independence property of hedges, i.e., for all $h, h', k, k' \in H$ we should find that if $h\delta < k\delta$ then $h'h\delta < k'k\delta$, where $\delta$ is a chain of hedges. In other words, $h'$ and $k'$ can not change the semantic ordering between $hx$ and $kx$. As we will show, this property carries over to the exponent. Note that this is different from Zadehs proposal of *CON* and *DIL* : They are commutative, i.e., *CON DIL* $A =$ *DIL CON* $A$.
- The set $H^* = \{\delta \mid \delta \text{ is a chain of hedges}\}$ of all membership modifiers should be mapped dense into the interval $(0, \infty)$. I.e. we should be able to approximate each value in $(0, \infty)$ to any required precision by an $exponent(\delta)$ such that $\delta \in H^*$. In case of $\beta$ approaches $0$, the correlative membership modifier may change membership degrees

of all individuals to 1, and in inverse case, if $\beta$ comes close to $+\infty$, then all membership degrees approximate 0 (See Figure 1.)

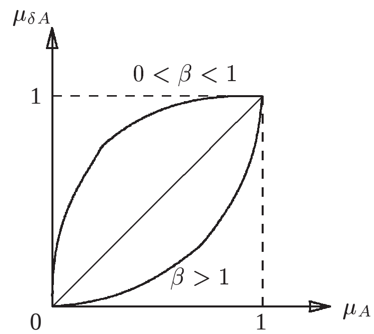


Figure 1: The membership modifier of $\delta A$.

Based on the above ideas, we favour in this paper an approach to construct membership modifiers in which the greatest positive modifier (e.g. "very") behaves like the "traditional" operator $CON$. I.e. we choose a value $upper > 1$ as exponent for $CON$, such that $very\ A = CON\ A = A^{upper}$, $very\ very\ A = CON\ CON\ A = A^{upper^2}$ and so forth. Likewise, we choose a value $0 < lower < 1$ as exponent for $DIL$, and treat the greatest negative modifier (e.g. "less") like $DIL$: $less\ A = DIL\ A = A^{lower}$, $very\ less\ A = DIL\ DIL\ A = A^{lower^2}$. The values $upper$ and $lower$ can be chosen arbitrarily in the range $upper > 1$ and $0 < lower < 1$ as parameters for the algorithm below to build a family of concept modifiers.

To simplify the computing process we assume that $H$ is linearly ordered, i.e., each two hedges are comparable, and restrict ourselves to hedge sets $H$ of an even cardinality $2p$, and sets of positive hedges and negative hedges of the same size $p$, i.e., we can represent $H$ as $H = \{h_1^-, h_2^-, ..., h_p^-, h_1^+, h_2^+, ..., h_p^+\}$, with positive hedges $h_1^+ < h_2^+ < ... < h_p^+$ and negative hedges $h_1^- < h_2^- < ... < h_p^-$. To avoid intermingling between positive or negative hedges, we restrict the mapping $sign$ such that for a given $h^*$ either all positive hedges $h_i^+$ increase the meaning of $h^*$ and all negative hedges $h_i^-$ decrease the meaning, i.e., $sign(h_i^+, h^*) = 1$ and $sign(h_i^-, h^*) = -1$, or the other way around, i.e., $sign(h_i^+, h^*) = -1$ and $sign(h_i^-, h^*) = 1$.

The algorithm of calculating $\beta$ for a chain $M = k_q k_{q-1} \ldots k_1$ of hedges is described in Figure 2. In the i-th step, $sig$ indicates that $k_i$ increases (by $sig = 1$) or decreases (by $sig = -1$) the value of $\beta_{i-1}$; $\beta_i$ will be the correlated value to $k_i k_{i-1} \ldots k_1$; and ( $lo_i, up_i$ ) defines the interval in which the exponent of $\delta k_i k_{i-1} \ldots k_1$ can be obtained, where $\delta$ is any chain of hedges. In case of most changing positive modifier (called by greatest positive modifier) like "very very ... very" we assign $\beta_i = upper^i$ and $up_i = upper^{i+1}$, to extend the interval for $\delta k_i k_{i-1} \ldots k_1$ to ( $lo_i, +\infty$ ). In this case is $mpos = 1$. If there exists only one $k_i$ in $k_q k_{q-1} \ldots k_1$ with $i \neq p$, then $mpos$ is changed to 0, and in this case it is not inspected any more. $mneg$ is dealt with in an inverse manner such that the exponent of $\delta k_i k_{i-1} \ldots k_1$ can receive values in ( $0, up_i$ ).

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Algorithm *exponent*

**Constants:** the values $upper > 1$, $0 < lower < 1$, the mapping $sign : H \times (H \cup \Omega) \to \{-1, 1\}$

**Input:** Concept modifier $M = k_q k_{q-1} \ldots k_1$ with $k_i \in H$ for all $i = \overline{1, q}$

**Output:** The exponent $\beta$

{
  $up_0 = upper$ ;  $lo_0 = lower$ ;  $\beta_0 = 1$ ;
  $mpos = 1$ ;  $mneg = 1$ ;  $sig = sign(k_1, A)$ ;
  for $i = 1$ to $q$ do {
    compute the index $j$ such that
      $k_i = h_j^-$ or $k_i = h_j^+$ ;
    if $i > 1$ then $sig = sig \times sign(k_i, k_{i-1})$ ;
    if $sig = 1$ then {
      $\beta_i = \beta_{i-1} + \frac{up_{i-1} - \beta_{i-1}}{2p} \times (2j - 1)$ ;
      $up_i = \beta_{i-1} + \frac{up_{i-1} - \beta_{i-1}}{2p} \times (2j)$ ;
      $lo_i = \beta_{i-1} + \frac{up_{i-1} - \beta_{i-1}}{2p} \times (2j - 2)$ ;
      if $mpos = 1$ and $j = p$ then {
        $\beta_i = upper^i$ ; $up_i = upper^{i+1}$ ; }
      else $mpos = 0$ ;
    }
    else {
      $\beta_i = \beta_{i-1} - \frac{\beta_{i-1} - lo_{i-1}}{2p} \times (2j - 1)$ ;
      $up_i = \beta_{i-1} - \frac{\beta_{i-1} - lo_{i-1}}{2p} \times (2j - 2)$ ;
      $lo_i = \beta_{i-1} - \frac{\beta_{i-1} - lo_{i-1}}{2p} \times (2j)$ ;
      if $mneg = 1$ and $j = p$ then {
        $\beta_i = lower^i$ ; $lo_i = lower^{i+1}$ ;}
      else $mneg = 0$ ;
    }
  }
  return( $\beta_q$ );
}

Figure 2: The algorithm to calculate the exponent $\beta$ of membership modifier $M$.

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In the following we will prove some properties of the algorithm. In the proofs we will often need to discuss what happens in the main loop, whose behaviour is different depending on the sequence $k_i k_{i-1} \ldots k_1$. To save space, we will define four cases that will be referred to in the proofs by their number. Let us denote $\Gamma_1 = \frac{up_{i-1} - \beta_{i-1}}{2p}$ and $\Gamma_2 = \frac{\beta_{i-1} - lo_{i-1}}{2p}$, by induction it is easy to examine that $\Gamma_1 > 0$ and $\Gamma_2 > 0$ for all $i$, and we have:

(I) $k_i$ is positive wrt $k_{i-1}...k_1$ and $k_i k_{i-1} \dots k_1$ is not the greatest positive modifier:

$$\begin{aligned} \beta_i &= \beta_{i-1} + \Gamma_1 \times (2j-1); \\ up_i &= \beta_{i-1} + \Gamma_1 \times (2j); \\ lo_i &= \beta_{i-1} + \Gamma_1 \times (2j-2). \end{aligned}$$

(II) $k_i$ is positive wrt $k_{i-1}...k_1$ and $k_i k_{i-1} \dots k_1$ is the greatest positive modifier:

$$\begin{aligned} \beta_i &= upper^{\,i}; \\ up_i &= upper^{\,i+1}; \\ lo_i &= \beta_{i-1} + \Gamma_1 \times (2p-2). \end{aligned}$$

(III) $k_i$ is negative wrt $k_{i-1}...k_1$ and $k_i k_{i-1} \dots k_1$ is not the greatest negative modifier:

$$\begin{aligned} \beta_i &= \beta_{i-1} - \Gamma_2 \times (2j-1); \\ up_i &= \beta_{i-1} - \Gamma_2 \times (2j-2); \\ lo_i &= \beta_{i-1} - \Gamma_2 \times (2j). \end{aligned}$$

(IV) $k_i$ is negative wrt $k_{i-1}...k_1$ and $k_i k_{i-1} \dots k_1$ is the greatest negative modifier:

$$\begin{aligned} \beta_i &= lower^{\,i}; \\ lo_i &= lower^{\,i+1}; \\ up_i &= \beta_{i-1} - \Gamma_2 \times (2p-2). \end{aligned}$$

Before turning to the properties of this algorithm we would like to illustrate it be means of an example.

**Example 1.**
Consider $H = \{very, mol\}$ with $p = 1$, $upper = 2$, $lower = 0.5$ and *sign* as follows:

| | $A$ | *very* | *mol* |
|---|---|---|---|
| *very* | 1 | 1 | −1 |
| *mol* | −1 | −1 | 1 |

Applying the algorithm we obtain:

| | *sig* | *mpos* | *mneg* | *lo* | $\beta$ | *up* |
|---|---|---|---|---|---|---|
| *very* | 1 | 1 | 0 | 1 | 2 | 4 |
| *mol* | −1 | 0 | 1 | 0.25 | 0.5 | 1 |
| *very very* | 1 | 1 | 0 | 2 | 4 | 8 |
| *mol very* | −1 | 0 | 0 | 1 | 1.5 | 2 |
| *very mol* | 1 | 0 | 0 | 0.5 | 0.75 | 1 |
| *mol mol* | −1 | 0 | 1 | 0.125 | 0.25 | 0.5 |

Let $M = k_q k_{q-1} \dots k_1$ be a concept modifier, the corresponding values are denoted by $lo_M$, $\beta_M$ and $up_M$. $\beta_M$ is $\beta_q$, the output of the Algorithm, and $lo_M$, $up_M$ are $lo_q$, $up_q$, which are calculated in the q-th step, respectively. For examination of the independence property we need the following lemma.

**Lemma 1.** For any concept modifier M we find: $lo_M < \beta_M < up_M$.

*Proof.* It is easy to prove in the cases (I), (III), (II): $\beta_M < up_M$ and (IV): $lo_M < \beta_M$ We have to consider the case (II), whether $lo_M < \beta_M$ is satisfied. The case (IV) is similar.

Because $2p-2 < 2p$ we also obtain $\frac{2p-2}{2p} < 1$. Therefore $lo_M = \beta_{q-1} + (up_{q-1} - \beta_{q-1})\frac{2p-2}{2p} < \beta_{q-1} + (up_{q-1} - \beta_{q-1}) \times 1 = \beta_{q-1} + up_{q-1} - \beta_{q-1} = upper^{\,q}$ because $k_{q-1} \dots k_1$ is also the greatest positive modifier of the length $q-1$. So $lo_M < \beta_M = upper^{\,q}$. □

**Lemma 2.** Let $k \in H$, then $lo_M \le \beta_{kM} \le up_M$.

*Proof.* It is easy to deduce the following constraints:
$lo_M < \beta_M \le lo_{kM} < \beta_{kM} < up_{kM} \le up_M$ in case (I)
$lo_M < \beta_M \le lo_{kM} < \beta_{kM} = up_M < up_{kM}$ in case (II)
$lo_M \le lo_{kM} < \beta_{kM} < up_{kM} \le \beta_M < up_M$ in case (III)
$lo_{kM} < lo_M = \beta_{kM} < up_{kM} \le \beta_M < up_M$ in case (IV)
Thus, $lo_M \le \beta_{kM} \le up_M$ holds in all cases. □

Therefore, we have the independence property of concept modifiers.

**Theorem 1.** Let $M_1 = \delta_1 h k_q \dots k_1$ and $M_2 = \delta_2 h' k_q \dots k_1$, with $h, h', k_i \in H$, $i = \overline{1,q}$, be two concept modifiers with the exponents $\beta_1, \beta_2$. Let $\beta, \beta'$ be the exponents of $hk_q \dots k_1$ and $h'k_q \dots k_1$. Then, for every $\delta_1, \delta_2$ we have that $\beta_1 < \beta_2$ holds if $\beta < \beta'$.

*Proof.* Let $M = k_q \dots k_1$. We support that $\delta_1 = h_r \dots h_1$. Since $\delta_1 hM$ is not the greatest positive modifier (because $\beta < \beta'$), we only need to pay attention to the cases (I), (III) and (IV). There are usually $\beta_{hM} < up_{hM} \le up_M$. Thus,
$\beta_1 = \beta_{h_r \dots h_1 hM} < up_{h_r \dots h_1 hM} \le up_{h_{r-1} \dots h_1 hM} \le \dots \le up_{h_1 hM} \le up_{hM}$
Similarly, we can obtain: $lo_{h'M} < \beta_2$.
Now, we examine that $up_{hM} \le lo_{h'M}$
According to the assumption, $\beta < \beta'$, so the next cases can happen:
1. $h$ and $h'$ are positive wrt $M$, with $h = h_j^+$, $h' = h_{j'}^+$ or $h = h_j^-$, $h' = h_{j'}^-$ and $j < j' \le p$, For $j < j'$, i.e. $j \le j' - 1$, or $2j \le 2j' - 2$, it follows

$$\beta_M + \frac{up_M - \beta_M}{2p}(2j) \le \beta_M + \frac{up_M - \beta_M}{2p}(2j' - 2),$$

or $up_{hM} \le lo_{h'M}$.
2. $h$ and $h'$ are negative wrt $M$: it is similar to the positive case.
3. $h$ is negative and $h'$ is positive wrt $M$: it is easy to prove because $up_{hM} \le \beta_M \le lo_{h'M}$
Thus, the theorem is completely proven. □

Now, we examine that our membership modifiers can change the membership degree of being an individual $d$ an element of a concept $A$ to any value in $[0,1]$, i.e. for $\langle d : A = n \rangle$ and giving $m \in [0,1]$, there exists a concept modifier $M$ with the exponent $\beta$, such that $\langle d : MA = n^\beta \rangle$ and $|n^\beta - m| < \epsilon$, with $\epsilon > 0$. We prove the following lemma.

**Lemma 3.** Let $M_q^* = \{M \mid length(M) = q\}$ be the set of all concept modifiers containing $q$ hedges, ( $lo_M, up_M$ ) be the intervall for exponents of $\delta M$ for every $\delta$. Then

$$\bigcup_{M \in M_q^*} [lo_M, up_M] = [lower^{q+1}, upper^{q+1}]$$

*Proof.* Let us denote $M = hM'$, with $h \in H$ and $M'$ contains $q-1$ hedges. By examining all four cases (I), (II), (III) and (IV) we obtain:

$$\bigcup_{h \in H} [lo_{hM'}, up_{hM'}] = \begin{cases} [lower^{q+1}, lower^{q-1}] \\ \quad \text{if } M' = \underline{M} \text{ is the} \\ \quad \text{greatest neg. modifier} \\ [upper^{q-1}, upper^{q+1}] \\ \quad \text{if } M' = \overline{M} \text{ is the} \\ \quad \text{greatest pos. modifier} \\ [lo_{M'}, up_{M'}] \\ \quad \text{otherwise} \end{cases}$$

Besides,

$$\bigcup_{M' \neq \underline{M} and M' \neq \overline{M}} [lo_{M'}, up_{M'}] = [lower^{q-1}, upper^{q-1}]$$

Thus,

$$\bigcup_{M \in M_q^*} [lo_M, up_M] = \bigcup_{M' \in M_{q-1}^*} [\bigcup_{h \in H} [lo_{hM'}, up_{hM'}]]$$

$= [lower^{q+1}, lower^{q-1}] \cup [lower^{q-1}, upper^{q-1}] \cup [upper^{q-1}, upper^{q+1}] = [lower^{q+1}, upper^{q+1}]$ □

**Theorem 2.** For $x \in [0, +\infty]$ and any $\epsilon > 0$, there exists a concept modifier $M$ such that $|\beta - x| < \epsilon$, where $\beta$ is the exponent of $M$.

*Proof.* For the giving $x$, we can choose $q$ such that $x \in [lower^{q+1}, upper^{q+1}]$. By Lemma 3 there exists $M \in M_q^*$ such that $x \in [lo_M, up_M]$. So, we have, $|\beta_M - x| < up_M - lo_M$. Since furthermore $h \in H$, it holds $|\beta_{hM} - x| < up_{hM} - lo_{hM} < up_M - lo_M$. And so on, this process continues until we obtain $\delta$ such that $|\beta_{\delta M} - x| < up_{\delta M} - lo_{\delta M} < \epsilon$ □

Hence, our exponential membership modifiers generate a family of modified concepts from a primitive concept $A$, and change the membership degree of an individual as an element of those. It follows, let $\langle d\!:\!A = n\rangle$ be a fuzzy assertion and $m \in [0,1]$, for any $\epsilon > 0$, there exists a concept modifier $M$ with the exponent $\beta$, such that $\langle d\!:\!MA = n^\beta\rangle$ and $|n^\beta - m| < \epsilon$. Furthermore, there is a subsumption relation between modified concepts.

**Theorem 3.** Let $C = MA$ and $D = M'A$ be two modified concepts from primitive concept $A$, and $\beta, \beta'$ are the exponents of $M, M'$. Then, $D$ subsumes $C$ iff $\beta \geq \beta'$.

*Proof.* $\Rightarrow$: $\beta \geq \beta'$ implies $x^\beta \leq x^{\beta'}$ for all $x \in [0,1]$, or $(A^{\mathcal{I}}(d))^\beta \leq (A^{\mathcal{I}}(d))^{\beta'}$ for all individuals $d$. It follows $(MA)^{\mathcal{I}}(d) \leq (M'A)^{\mathcal{I}}(d)$ for all $d$, or $C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$. Also, $D$ fuzzy subsumes $C$.

$\Leftarrow$: for $(MA)^{\mathcal{I}}(d) \leq (M'A)^{\mathcal{I}}(d)$ for all $d$, or $(A^{\mathcal{I}}(d))^\beta \leq (A^{\mathcal{I}}(d))^{\beta'}$, we have $\beta \geq \beta'$. □

Now we are ready to turn our attention to decision procedures.

## 4. Decision procedure

In this section we present a decision procedure for unsatisfiability in $\mathcal{ALC}_{FH}$. As usual, other problems can be reduced to an unsatisfiability problem, e.g. the satisfiability problem:

$$\Sigma \models \langle (w_1, w_2)\!:\!R \circ n\rangle$$
$$\text{iff } \Sigma \cup \langle (w_1, w_2)\!:\!R \bullet n\rangle \text{ is unsatisfiable,}$$

$$\Sigma \models \langle w\!:\!C \circ n\rangle$$
$$\text{iff } \quad \Sigma \cup \langle w\!:\!C \bullet n\rangle \text{ is unsatisfiable,}$$

where $\circ$ is one of $>, \geq, \leq, <$, and $\bullet$ is the negated operator $\leq, <, >, \geq$, respectively. The assertion of type $\langle \alpha = n\rangle$ is not explicitly considered here, since it can easily be reduced to assertions using $\leq$ and $\geq$:

$$\Sigma \models \langle \alpha = n\rangle \quad \text{iff}$$
$$\Sigma \models \langle \alpha \leq n\rangle \quad \text{and} \quad \Sigma \models \langle \alpha \geq n\rangle \ ,$$

resp.

$$\Sigma, \langle \alpha = n\rangle \models \psi \quad \text{iff}$$
$$\Sigma, \langle \alpha \leq n\rangle, \langle \alpha \geq n\rangle \models \psi \ .$$

Due to lack of space we do not discuss the subsumption problem in this paper, however we conjecture that it can be reduced to the satisfiability problem by a process similar to knowledge base expansion [7], as in the closely related fuzzy $\mathcal{ALC}$ [8].

Our unsatisfiability decision procedure closely follows [8], but introduces a new set of rules for the handling of concept modifiers. Starting from a set $S$ of fuzzy constraints, we apply propagation rules to add “simpler” constraints preserving the satisfiability. This process continues until we either find some contradictory constraints (a *clash*), thus proving unsatisfiability, or if no more rules are applicable. In the latter case we can construct a model from the completed set of constraints. Due to the non–deterministic nature of some of the rules, unsatisfiability has to be proven for possible all choices in the application of the rules. Thus, in general backtracking will be necessary to prove unsatisfiability.

$S$ contains a *clash* iff it contains either one of the unsatisfiable fuzzy constraints $\langle \bot \geq n\rangle$ where $n > 0$, $\langle \top \leq n\rangle$ where $n < 1$, $\langle \alpha < 0\rangle$, $\langle \alpha > 1\rangle$, $\langle \bot > n\rangle$, or $\langle \top < n\rangle$, or $S$ contains a conjugated pair of fuzzy constraints as in Table 3.

| | $\langle \alpha < m \rangle$ | $\langle \alpha \leq m \rangle$ |
|---|---|---|
| $\langle \alpha \geq n \rangle$ | $n \geq m$ | $n > m$ |
| $\langle \alpha > n \rangle$ | $n \geq m$ | $n \geq m$ |

Table 3: The conditions, under which a row-column pair of fuzzy constraints is conjugated.

Table 4 contains the rules for of calculus. We have augmented the rules from [8] by the rules $(M_>), (M_\geq)$, $(M_\leq)$ and $(M_<)$ to handle the concept modifiers. These propagation rules have the form $\Phi \rightarrow \Psi$ if $\Gamma$, where $\Phi$ and $\Psi$ are sets of fuzzy constraints and $\Gamma$ is a condition. A rule can be applied to a set $S$ of fuzzy constraints if the condition $\Gamma$ holds wrt $S$, $\Phi \subseteq S$ and $\Psi$ contains constraints not yet contained in $S$. As the result of the application the constraints in $\Psi$ are added to $S$. An exception to this application rule are the non-deterministic rules $(\sqcup_\geq), (\sqcup_>), (\sqcap_\leq)$ and $(\sqcap_<)$, where $\Psi$ is of the form $\Psi_1 \mid \Psi_2$. They can be applied if $\Gamma$ holds wrt $S$, $\Phi \subseteq S$ and neither $\Psi_1$ nor $\Psi_2$ is a subset of $S$. The result of their application is non-deterministically either the addition of $\Psi_1$ or the addition of $\Psi_2$ to $S$.

A *completion* of a finite set of constraints $S$ is the result of the application of the rules in Table 4 starting with $S$ until no more rules can be applied. The calculus has the *termination property*, i.e. the rules can only be applied finitely often to a finite set of constraints: Since the sequents the rules add to the set have smaller term sizes than the antecedents, and each rule can be applied only once (or finitely often in the case of $(\forall_\geq)$ and $(\exists_\leq)$) to the same antecedent, there can be no infinite chain of rule applications.

**Example 2.** Let us consider the following knowledge base $\Sigma$:

"If a customer is very interested in a property, he wants a product $p$ that has that property" holds to a degree more than 0.9:

(1) $\langle p : \neg(\exists \text{attr}, \textit{very}\, \text{int}) \sqcup \text{wants} \geq 0.9 \rangle$

"The customer is interested in technics" to $\geq 0.8$:

(2) $\langle \textit{tech} : \text{int} \geq 0.8 \rangle$

"The product $p$ has attribute technics" to $\geq 0.7$:

(3) $\langle (p, \textit{tech}) : \text{attr} \geq 0.7 \rangle$ .

We want to prove that $\Sigma$ entails that the customer wants product $p$ to a degree of at least 0.9. That is, we have to prove the unsatisfiability of $\Sigma$ together with the complement of the claim:

(4) $\langle p : \text{wants} < 0.9 \rangle$

Application of rule $(\sqcup_\geq)$ is non–deterministic. We have to consider the case

(5) $\langle p : \text{wants} \geq 0.9 \rangle$ ,

$(\neg_\geq)$ $\langle w : \neg C \geq n \rangle \rightarrow \langle w : C \leq 1 - n \rangle$

$(\neg_\leq)$ $\langle w : \neg C \leq n \rangle \rightarrow \langle w : C \geq 1 - n \rangle$

$(\sqcap_\geq)$ $\langle w : C \sqcap D \geq n \rangle \rightarrow \langle w : C \geq n \rangle, \langle w : D \geq n \rangle$

$(\sqcup_\leq)$ $\langle w : C \sqcup D \leq n \rangle \rightarrow \langle w : C \leq n \rangle, \langle w : D \leq n \rangle$

$(\sqcup_\geq)$ $\langle w : C \sqcup D \geq n \rangle \rightarrow \langle w : C \geq n \rangle \mid \langle w : D \geq n \rangle$

$(\sqcap_\leq)$ $\langle w : C \sqcap D \leq n \rangle \rightarrow \langle w : C \leq n \rangle \mid \langle w : D \leq n \rangle$

$(M_\geq)$ $\langle w : MA \geq n \rangle \rightarrow \langle w : A \geq n^{\frac{1}{\beta}} \rangle$ if $\beta =$ *exponent* $(M)$

$(M_\leq)$ $\langle w : MA \leq n \rangle \rightarrow \langle w : A \leq n^{\frac{1}{\beta}} \rangle$ if $\beta =$ *exponent* $(M)$

$(M_>)$ $\langle w : MA > n \rangle \rightarrow \langle w : A > n^{\frac{1}{\beta}} \rangle$ if $\beta =$ *exponent* $(M)$

$(M_<)$ $\langle w : MA < n \rangle \rightarrow \langle w : A < n^{\frac{1}{\beta}} \rangle$ if $\beta =$ *exponent* $(M)$

$(\forall_\geq)$ $\langle w_1 : \forall R.C \geq n \rangle, \psi \rightarrow \langle w_2 : C \geq n \rangle$ if $\psi$ is conjugated to $\langle (w_1, w_2) : R \leq 1 - n \rangle \in S$

$(\exists_\leq)$ $\langle w_1 : \exists R.C \leq n \rangle, \psi \rightarrow \langle w_2 : C \leq n \rangle$ if $\psi$ is conjugated to $\langle (w_1, w_2) : R \leq n \rangle$

$(\exists_\geq)$ $\langle w : \exists R.C \geq n \rangle \rightarrow \langle (w, x) : R \geq n \rangle, \langle x : C \geq n \rangle$ if $x$ is a new variable and there is no $w'$ such that both $\langle (w, w') : R \geq n \rangle$ and $\langle w' : C \geq n \rangle$ are already in the constraint set

$(\forall_\leq)$ $\langle w : \forall R.C \leq n \rangle \rightarrow \langle (w, x) : R \geq 1 - n \rangle, \langle x : C \leq n \rangle$ if $x$ is a new variable and there is no $w'$ such that both $\langle (w, w') : R \geq 1 - n \rangle$ and $\langle w' : C \leq n \rangle$ are already in the constraint set

Table 4: The rules of the decision procedure. In addition to the presented rules there are rules $(\neg_>)$, $(\neg_<)$, $(\sqcap_>) \ldots (\forall_<)$ for the strict relations. These can easily be obtained from the rules above by replacing $\geq$ by $>$ and $\leq$ by $<$.

which immediately yields a clash with (4). The other case yields

(5') $\langle p:\neg(\exists \text{attr}, \textit{very}\, \text{int}) \geq 0.9\rangle$ ,

(6') $\langle p:\exists \text{attr}, \textit{very}\, \text{int} \leq 0.1\rangle$ ,

Since (3) is conjucated to $\langle (p, \textit{tech}) : \text{attr} \leq 0.1\rangle$, rule $(\exists_{\leq})$ yields:

(7') $\langle \textit{tech}: \textit{very}\, \text{int} \leq 0.1\rangle$

Application of $(M_{\leq})$ with $\textit{exponent}(\textit{very}) = 2$ gives

(8') $\langle \textit{tech}: \text{int} \leq 0.316\rangle$ ,

which clashes with (2). Thus, there is no clash–free completion of $\Sigma \cup \{(4)\}$. Hence, it is unsatisfiable, and consequently $\Sigma \models \langle p: \text{wants} \geq 0.9\rangle$.

The discussed rules give us a decision procedure for $\mathcal{ALC}_{FH}$, because we can construct a model for a set $S$ of constraints from a clash–free completion of $S$ or prove that there is no model for $S$ if there os no clash–free completion.

**Theorem 4.** A finite set of fuzzy constraints $S$ is satisfiable iff there exists a clash free completion of $S$.

*Proof.*

$\Rightarrow$**:** By case analysis it is easily verified that the rules are sound, i.e. if we apply a rule to a satisfiable constraint–set $S_1$, the result $S_2$ is satisfiable as well, and thus, clash–free. Let us just give two examples for the rules $(\exists_{\leq})$ and $(M_{\geq})$.

$(\exists_{\leq})$**:** Assume $(\exists_{\leq})$ is applicable, i.e. $S_1$ contains $\langle w_1 : \exists R.C \leq n\rangle$ and $\langle (w_1, w_2) : R \geq n\rangle$ for some $w_1, w_2$ and $1 \geq m > n \geq 0$. [5] Since $S_1$ is satisfiable, there is an interpretation $\mathcal{I}$ that satisfies both $\langle w_1 : \exists R.C \leq n\rangle$ and $\langle (w_1, w_2) : R \geq n\rangle$. If $\langle w_2 : C \leq m\rangle$ was not satisfied by $\mathcal{I}$, we would have $\min\{R^{\mathcal{I}}(w_1^{\mathcal{I}}, w_2^{\mathcal{I}}), C^{\mathcal{I}}(w_2^{\mathcal{I}})\} = m > n$, and thus $(\exists R.C)^{\mathcal{I}} > n$ according to Table 2. This contradicts our assumption, i.e. $\langle w_2 : C \leq m\rangle$ is satisfied by $\mathcal{I}$, and thus $S_2$.

$(M_{\geq})$**:** Assume $(M_{\geq})$ is applicable, i.e. $S_1$ contains $\langle w : MA \geq n\rangle$ for some $W$, $n$, and the concept modifier $M$ applied to the primitive concept $A$. Since $S_1$ is satisfiable, there is an interpretation $\mathcal{I}$ that satisfies $\langle w : MA \geq n\rangle$, and thus $(MA)^{\mathcal{I}}(w^{\mathcal{I}}) = (A)^{\mathcal{I}}(w^{\mathcal{I}})^{\beta} \geq n$. Let $\beta = \textit{exponent}(M)$. Since $\beta > 0$ and $n \geq 0$ we can conclude $(A)^{\mathcal{I}}(w^{\mathcal{I}}) \geq n^{\frac{1}{\beta}}$. Thus, $S_2$ is satisfied by $\mathcal{I}$ as well.

$\Leftarrow$**:** Assume $S'$ is a clash–free completion of $S$. We will now construct a model for the fuzzy constraints of $S'$ that only contain primitive concepts or roles, and prove that it is a model of $S'$, and thus $S$, as well.

[5] We skip the similar other case $\langle w_2 : C > m\rangle$ with $m \geq n$ for the clash, here.

Consider the interpretation $\mathcal{I}$ such that the domain $\Delta^{\mathcal{I}}$ is the set of objects appearing in $S'$, $w^{\mathcal{I}} = w$ for all $w \in \Delta^{\mathcal{I}}$, and that maps the primitive concepts and the roles to the median of the lowest upper bound and the greatest lower bound given in $S'$, inclusive the implicit constraints 0 and 1:

$$A^{\mathcal{I}}(w^{\mathcal{I}}) = \frac{1}{2}\max\left(\begin{array}{c}\{n \mid \langle w : A > n\rangle \in S'\}\cup\\ \{n \mid \langle w : A \geq n\rangle \in S'\} \cup \{0\}\end{array}\right) + \frac{1}{2}\min\left(\begin{array}{c}\{n \mid \langle w : A < n\rangle \in S'\}\cup\\ \{n \mid \langle w : A \leq n\rangle \in S'\} \cup \{1\})\end{array}\right)$$

$$R^{\mathcal{I}}(w_1^{\mathcal{I}}, w_2^{\mathcal{I}}) = \frac{1}{2}\max\left(\begin{array}{c}\{n \mid \langle (w_1, w_2) : R > n\rangle \in S'\}\cup\\ \{n \mid \langle (w_1, w_2) : R \geq n\rangle \in S'\}\cup\\ \{0\}\end{array}\right) + \frac{1}{2}\min\left(\begin{array}{c}\{n \mid \langle (w_1, w_2) : R < n\rangle \in S'\}\cup\\ \{n \mid \langle (w_1, w_2) : R \leq n\rangle \in S'\}\cup\\ \{1\}\end{array}\right)$$

It can easily be verified that this interpretation satisfies all constraints for primitive concepts and roles given in $S'$ if $S'$ is clash–free. The satisfaction of the other fuzzy constraints in $S'$ are shown by induction on the term structure of the $\mathcal{ALC}_{FH}$ -formula in the constraints. Again, we just present two cases for space reasons.

**Case** $\langle w : \forall R.C < n\rangle$ Since $S'$ is complete, there is a $w'$ such that $\langle (w, x) : R > 1 - n\rangle$ and $\langle x : C < n\rangle$ are in $S'$ and are satisfied by $\mathcal{I}$ by induction assumption.[6] Thus, there is a $w' \in \Delta^{\mathcal{I}}$ such that $1 - R^{\mathcal{I}}(w^{\mathcal{I}}, w'^{\mathcal{I}}) < n$ and $C^{\mathcal{I}}(w'^{\mathcal{I}}) < n$, and hence $(\forall R.C)^{\mathcal{I}} < n$.

**Case** $\langle w : MA > n\rangle$**:** Since $S'$ is complete, $\langle w : A > n^{\frac{1}{\beta}}\rangle$ with $\beta = \textit{exponent}(M)$ is in $S'$ and is satisfied by $\mathcal{I}$ by induction assumption. Thus, $A^{\mathcal{I}}(w^{\mathcal{I}}) > n^{\frac{1}{\beta}}$, and since $\beta > 0$ we have that $(MA)^{\mathcal{I}}(w^{\mathcal{I}}) > n$.

□

## 5. Truth Bounds and Expressivity

In practice it is often important to determine bounds for the truth values at which fuzzy constraints are entailed or can be satisfied. For instance, for a knowledge base $\Sigma$ the lowest value of $n$ (called the *lower truth bound*) such that $\Sigma \models \langle k : \text{technical} \geq n\rangle$ might determine how much the customer $k$ is interested in technical aspects. [8] proves for Fuzzy-$\mathcal{ALC}$ that $n$ is one of the values occurring in $\Sigma$, their complements, or one of $\{0, 0.5, 1\}$. The following example shows that this is not true for our

[6] $w'$ is either the new variable indtroduced by the application of $(\forall_{<})$, or the $w'$ that prevented the execution of $(\forall_{<})$.

language. Thus, the binary search in that set suggested in [8] does not suffice to calculate the truth values in $\mathcal{ALC}_{FH}$ . This also shows that our language is strictly more expressive than Fuzzy-$\mathcal{ALC}$ .

**Example 3.** Consider the satisfiability problem

(9) $\emptyset \models \langle \text{technical} \sqcap \neg \textit{very}\ \text{technical} \geq n \rangle$ .

As the reader can easily verify, this is satisfiable whenever $n$ is less than $1 - x$, where $x$ is the solution for $1 - x^2 = x$ and $\beta = \textit{exponent}(very)$. If we choose the hedges as in Example 1, we have $\beta = 2$, and thus (9) is satisfiable iff $n \geq 0.618\ldots$.

However, one can easily approximate the truth bounds to any given precision by a binary search in the interval $[0, 1]$. E.g., starting from $[0, 1]$, an interval $[l, u]$ for the lower truth bound for $n$ in $\Sigma \models k : C \geq n$ can successively be halved to $[l, \frac{u+l}{2}]$ or $[\frac{u+l}{2}, u]$ by checking whether $\Sigma \models k : C \geq \frac{u+l}{2}$ is satisfiable. It seems likely that the application of constraint propagation techniques can improve this process very much.

## 6. Complexity

Because the language Fuzzy $\mathcal{ALC}$ defined in [8] is a subset of $\mathcal{ALC}_{FH}$ and we use a similar decision procedure, the PSPACE-hardness of deciding entailment for Fuzzy $\mathcal{ALC}$ carries over to our language. Another result that carries over is, that the rules for the quantifiers $\exists$ and $\forall$ can lead to exponential space requirements in a proof. [8] alleviates this problem through the introduction of so called *trace rules* for the quantors, and we believe this is possible for our case as well.

## 7. Discussion

Description logics are a widely accepted formalism for representing and reasoning about a terminology based on crisp concepts and relations. Fuzzy logic is a widely accepted formalism for representing and reasoning about non-crisp concepts and relations. Surprisingly, there have been only few attempts to combine the advantages of both within one system that we are aware of, viz. [9] and [8].

In this paper we have presented the fuzzy description logic with hedges $\mathcal{ALC}_{FH}$ . It is a conservative extension of Fuzzy-$\mathcal{ALC}$ defined by Umberto Straccia in [8]. In [8] the classical description logic $\mathcal{ALC}$ is extended by interpreting concepts and relations as fuzzy sets, the correspondence of the fuzzy semantics and the semantics of the equivalent “crisp” $\mathcal{ALC}$ is discussed and decision procedures for the entailment and subsumption problem have been defined. We extend the fuzzy description logic given there by concept manipulators whose semantics is based on hedge algebras.

Christopher B. Tresp an Ralf Molitor [9] extend $\mathcal{ALC}$ with membership manipulators, i.e., functions that map the set $[0, 1]$ of fuzzy truth values to $[0, 1]$. These can be applied to the fuzzy sets representing concepts in order to construct new concepts. The membership manipulators are restricted to triangular functions on $[0, 1]$. Tresp and Molitor also define a constraint propagation based method for computing the degree of subsumption between two concepts. However, their consequence relation is counter-intuitive to us in some cases. In [9] a concept $D$ is subsumed by a concept $C$ by the minimum degree of $D$ being satisfied in interpretations that map $C$ to $1$. Consequently, they can draw arbitrary conclusions from a premise which is never satisfied to degree $1$, but, say, only to $0.6$. Such a concept is discussed in Example 3. In $\mathcal{ALC}_{FH}$ membership manipulators are hedges and the subsumption of modified concepts is based–as we believe quite intuitively–on the “less than or equal” relation of the corresponding $\beta$-values as shown in Theorem 8. On the other hand, we can handle only primitive concepts, whereas Tresp and Molitor deal with concepts in general.

We hope that this paper is the starting point of a fruitful combination of description logic, fuzzy logic and hedge algebras based on the idea of manipulating concepts by hedges. In deed, many interesting problems are yet to be solved. Hedges were only applied to primitive concepts. The application of concept modifiers to non–primitive concept seems possible, but it leads to some counter–intuitive cases where e.g. the expression *very mol* $A$ can have two different meanings: *very* (*mol* $A$) or (*very mol*) $A$ . In this paper we have only discussed the unsatisfiability problem. Although we claim that the subsumption problem is also decidable and can be dealt with by extending the corresponding algorithm in [8], it remains to turn this claim into a theorem. Likewise, we believe that trace rules introduced in [8] for Fuzzy-$\mathcal{ALC}$ can be extended to deal with $\mathcal{ALC}_{FH}$ , but, again, this remains to be rigorously shown. Finding a good real-world application is vital for the success of the proposed method. We believe that the field of intelligent user-adaptive e-commerce provides such examples.

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